THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW11 Solution

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1. (P.286 Q3)

   Let $\epsilon > 0$ be given, since $\sum_n c_n \sin nx$ converges uniformly on $\mathbb{R}$, by Cauchy Criterion (9.4.5), there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x \in \mathbb{R}$

$$|c_n \sin nx + ... + c_{2n} \sin 2nx| < \epsilon$$

   Now fix $n \geq N$, choose $x = \dfrac{\pi}{6n}$, then for all $k \in \mathbb{N}$ such that $n \leq k \leq 2n$, $\dfrac{1}{2} \leq \sin kx \leq 1$. Since (c_n) is a sequence of positive decreading he above inequality becomes

$$\begin{aligned} \epsilon &> |c_n \sin nx + ... + c_{2n} \sin 2nx| \\ &= c_n \sin nx + ... + c_{2n} \sin 2nx \\ &\geq \frac{1}{2}(c_n + ... + c_{2n}) \\ &\geq \frac{n+1}{2} c_{2n} \end{aligned}$$

   Therefore, $2(n+1)c_{2n} < 4\epsilon$. Now we claim that for all $m \geq 2N+1$, $|mc_m| < 4\epsilon$

   Case I: m is even: then $m = 2n$ for some $n \geq N$, then by above inequality,

$$|mc_m| = 2nc_{2n} < 2(n+1)c_{2n} < 4\epsilon$$

   Case II: m is odd: then $m = 2n+1$ for some $n \geq N$, then by above inequality,

$$|mc_m| = (2n+1)c_{2n+1} < 2(n+1)c_{2n} < 4\epsilon$$

   Therefore, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq 2N$, $|mc_m| < 4\epsilon$, i.e.

$$\lim_n nc_n = 0$$

2. (P.286 Q6b)

   Due to the result of Q5 of Exercises 9.4, it suffices to compute $\displaystyle\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$:

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n^\alpha}{n!} \cdot \frac{(n+1)!}{(n+1)^\alpha} = \frac{n+1}{(1+\frac{1}{n})^\alpha}$$

Therefore, $\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{n+1}{(1+\frac{1}{n})^\alpha}=+\infty$, and hence the radius of convergence is $R=+\infty$.

3. (P.286 Q6c)

   Due to the result of Q5 of Exercises 9.4, it suffices to compute $\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}$:

$$\frac{|a_n|}{|a_{n+1}|}=\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^n}=\frac{1}{(1+\frac{1}{n})^n}$$

   Therefore, $\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}$, and hence the radius of convergence is $R=\frac{1}{e}$.

4. (P.286 Q6f)

   We compute $\rho=\lim_{n\to\infty}(|a_n|)^{\frac{1}{n}}$ directly:

$$(|a_n|)^{\frac{1}{n}}=(n^{-\sqrt{n}})^{\frac{1}{n}}=\frac{1}{n^{\frac{1}{\sqrt{n}}}}$$

   To compute $\lim_{n\to\infty}n^{\frac{1}{\sqrt{n}}}$, it suffices to compute $\lim_{x\to\infty}x^{\frac{1}{\sqrt{x}}}=\lim_{x\to\infty}e^{\frac{\ln x}{\sqrt{x}}}$.

   In view of this, we compute $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$. By L'Hospital's Rule,

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

   Therefore, $\lim_{x\to\infty}x^{\frac{1}{\sqrt{x}}}=e^0=1$, and hence by sequential criterion, $\rho=\lim_{n\to\infty}n^{\frac{1}{\sqrt{n}}}=\lim_{x\to\infty}x^{\frac{1}{\sqrt{x}}}=1$, and hence the radius of convergence is $R=\frac{1}{\rho}=1$.